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On the plant leaf's boundary, 'jupe à godets' and conformal embeddings

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Abstract

The stable profile of the boundary of a plant's leaf fluctuating in the direction transverse to the leaf's surface is described in the framework of a model called a 'surface à godets' (SG). It is shown that the information on the profile is encoded in the Jacobian of a conformal mapping (the coefficient of deformation) corresponding to an isometric embedding of a uniform Cayley tree into the 3D Euclidean space. The geometric characteristics of the leaf's boundary (such as the perimeter and the height) are calculated. In addition, a symbolic language allowing us to investigate the statistical properties of a SG with annealed random defects of the curvature of density q is developed. It is found that, at $q = 1$, the surface exhibits a phase transition with the critical exponent $\alpha = \frac{1}{2}$ from the exponentially growing to the flat structure.

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1. Introduction

The subject of this paper was inspired by the following questions raised by V E Zakharov in a private conversation about two years ago: (1) 'What are the geometrical reasons for the boundary of plant leaves to fluctuate in the direction transverse to the leaf's surface?' and (2) 'How to describe the corresponding stable profile?' The answer to question (1) came almost immediately.

It is reasonable to imagine that cells located near the leaf's boundary proliferate more actively than ones in the bulk of the leaf. One possible reason might be purely geometric: the peripheral cells are not completely enclosed by the surrounding media and hence have more available space for growth than the cells inside the leaf's body[3]. The corresponding

[3] Despite this explanation seeming rather natural, some biological justifications for the hypothesis expressed would be worthwhile.

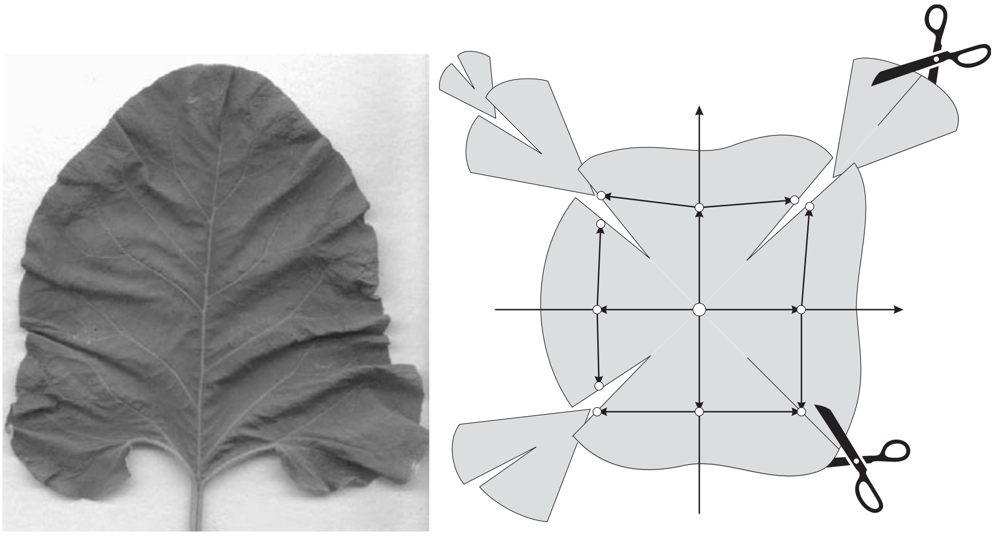


Figure 1. (*a*) Picture of a burdock leaf (*Arctium lappa*). (*b*) The surface à godets isometrically covered by a four-branching Cayley tree.

picture of the burdock leaf (*Arctium lappa*) is reproduced in figure 1(*a*). This phenomenon can be observed for various species such as most types of lettuce (*Lactuca sativa*) or spinach (*Spinacea oleracea*). The schematic model of the suggested geometric origin of the leaf's folding is shown in figure 1(*b*) (see the explanations below).

At the same time question (2) remained unanswered, and in this paper we discuss some possible ways of finding a solution to this problem. Relying on the growth mechanism of a plant's leaf, as conjectured above, one can propose the following naive construction schematically shown in figure 1(*b*). Take some 'elementary' bounded domain of a flat surface, make finite radial cuts and insert in these cuts flat triangles, modelling the area newly generated by peripheral cells. The resulting surface, obtained by putting back together all the elementary domains and added extra triangles, is no longer flat. Continue the process recursively, i.e. make cuts in the new surface, insert extra flat triangles, and so on. One obtains in this way a surface whose perimeter and area grow exponentially with the radius. Several examples can be found in [1].

Apparently a similar problem was discussed for the first time by the Russian mathematician P L Tchebychef in his talk 'On the cut of clothes' in Paris on 28 August 1878, but apparently unpublished (some traces of this talk can be found in [2]). Among the modern developments of this subject one can mention the paper by Bowers [3] proving in 1997 the theorem on quasi-isometric embedding of a uniform binary tree into a negatively curved space, as well as the contribution by Duval and Tajine [4] devoted to isometric embeddings of trees into metric spaces for fractal descriptions.

Our following deliberation is based on two suppositions: (i) the plant's leaf has infinitesimal thickness without any surface tension, and (ii) the activity of the boundary cells is independent of the size of the leaf.

The surface constructed above is called 'hyperbolic' [5]. Anyone who pays attention to the tendencies in fashion recognizes in this recursive construction the so-called 'jupe à godets'. Later on we shall call such surfaces the 'surfaces à godets' (SG). Despite this rather transparent geometrical image, the problem under consideration is still too vague. Let us formulate it in more rigorous terms, which allows for its mathematical analysis.

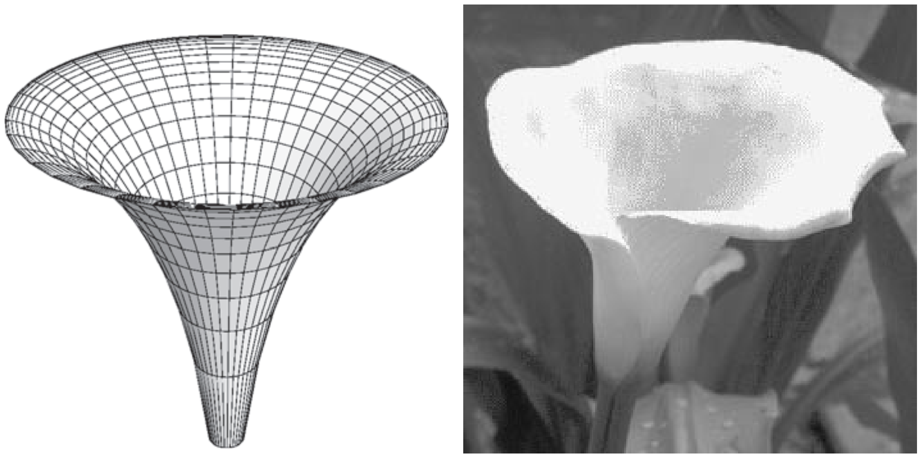


Figure 2. (*a*) Pseudosphere. (*b*) Picture of a calla lilly.

Let us cover the SG by a natural 'lattice'. The construction of this 'lattice' is as follows. Take a four-branching Cayley tree[4]. It is well known that any regular Cayley tree, as an exponentially growing structure, cannot be isometrically embedded in any space with a flat metric. One could expect that the 'SG', being by construction an exponentially growing (i.e. 'hyperbolic') structure, admits Cayley trees as possible discretizations. The reason for such a choice is based on the fact that the circumference $P(k)$ of the Cayley tree (i.e. the number of outer vertices located at the distance k from the tree root) grows as $P(k) = z \times (z-1)^{k-1}$, where z is the coordinational number of the Cayley tree ($z = 4$ for a four-branching tree). We shall assume that Cayley trees can cover the 'SG' *isometrically*[5], i.e. without gaps and self-intersections, preserving angles and distances—see figures 1(*b*) and 6(*a*). Thus, our further aim consists in describing the relief of the 'SG' in three-dimensional Euclidean space, under the condition that a regular four-branching Cayley tree is isometrically embedded in this surface.

Comment. Let us stress that in our paper we are interested in the embedding of open (i.e. unbounded) surfaces only. From many textbooks on non-Euclidean geometry we know that exponentially growing surfaces *with a boundary* can be isometrically embedded in a Euclidean space. The surface of revolution of the so-called *tractrix* around its asymptote is a famous example of such an embedding [5]—see figure 2(*a*). In a 3D Euclidean space x, y, z the pseudosphere (PS) is parametrized by the following equations:

$$\begin{aligned} x &= \cos u \, \sin v \\ y &= \sin u \, \sin v \\ z &= \cos v + \log \tan \frac{v}{2} \end{aligned}$$

where $0 \leqslant u \leqslant 2\pi$ and $0 < v \leqslant \frac{\pi}{2}$.

One sees from figure 2(*a*) that the surface of the PS has no wrinkles and, as it is bounded, it contains a *finite part* of a Cayley tree as an isometric embedding. We suspect that the pseudospherical structure can also be realized in the plant world. The picture of the calla lilly reproduced in figure 2(*b*) supports our claim.

[4] The example of a four-branching Cayley tree is considered for simplicity. In principle, one can deal with any regular hyperbolic lattice. Some other examples are discussed at length in section 3.

[5] For example, the rectangular lattice shown in figure 6(*b*) isometrically covers the Euclidean plane.

From our discussion, we conjecture that, in the natural world, both the SG and the PS have equal rights to exist. From the biological point of view there is no difference between SG and PS—in both cases the same mechanism for cell proliferation is assumed. However, from the topological point of view the crucial point is the initial structure of the growing surface: if in the initial phase the growing structure is'pipe-like', then one can expect the PS formation, while if the initial surface is almost flat and all circumferential cells are independently growing, we ultimately arrive at a SG.

After this comment we turn to the question of embedding isometrically a four-branching *infinite* Cayley tree into a 2-manifold ('SG'), viewed as an open surface in the 3D Euclidean space.

2. Isometric embedding of a 'surface à godets' into the 3D Euclidean space

2.1. The model

Take a zero-angled rectangle $A_\zeta B_\zeta A'_\zeta C_\zeta$ bounded by arcs and lying in the unit disc $|\zeta| < 1$ as shown in figure 3(*d*). Make reflections of the interior of the rectangle with respect to its sides $A_\zeta B_\zeta$, $B_\zeta A'_\zeta$, $A'_\zeta C_\zeta$, $C_\zeta A_\zeta$ and obtain a new generation of vertices—the new images of the initial zero-angled rectangle [6]. For example, the reflection with respect to the arc $A_\zeta C_\zeta$ (which by definition remains unchanged) transforms $A'_\zeta \to A''_\zeta$ and $B_\zeta \to B'_\zeta$.

Proceeding recursively with reflections, one isometrically tessellates the disc $|\zeta| < 1$ with all images of the rectangle $A_\zeta B_\zeta A'_\zeta C_\zeta$. If one connects the centres of the neighbouring (i.e. obtained by successive reflections) rectangles, one gets a four-branching Cayley tree isometrically embedded into the unit disc endowed with the Poincaré metric [7] defined as follows:

$$\mathrm{d}s^2 = \frac{\mathrm{d}\zeta\,\mathrm{d}\zeta^*}{(1-\zeta\,\zeta^*)^2}$$

where $\mathrm{d}s$ is the differential length and the variables ζ and ζ^* are complex conjugated.

It is known that the tessellation of the Poincaré disc by circular zero-angled rectangles is uniform in a surface of a two-dimensional hyperboloid obtained by stereographic projection from the unit disc [5]. The open 2-hyperboloid is naturally embedded into a 3D space with a *Minkovski metric*, i.e. with a metric tensor of signature {+1, +1, −1}. However, the problem of uniform embedding of an open 2-hyperboloid into a 3D space with a *Euclidean metric* deserves special attention and exactly coincides with the main aim of our work—embedding a four-branching Cayley tree (isometrically covering the unit disc with a Poincaré metric) into a 3D Euclidean space. Such an embedding is realized by a transform $z = z(\zeta)$ which conformally maps a flat square $A_z B_z A'_z C_z$ in the complex plane z to a circular zero-angled rectangle $A_\zeta B_\zeta A'_\zeta C_\zeta$ in the unit disc $|\zeta| < 1$ (see figure 3). The relief of the corresponding surface is encoded in the so-called *coefficient of deformation* $J(\zeta) = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2$ coinciding with the Jacobian of the conformal transform $z(\zeta)$:

$$J(\zeta) = |z'(\zeta)|^2 = \begin{vmatrix} \dfrac{\partial\,\mathrm{Re}\,z}{\partial\,\mathrm{Re}\,\zeta} & \dfrac{\partial\,\mathrm{Re}\,z}{\partial\,\mathrm{Im}\,\zeta} \\ \dfrac{\partial\,\mathrm{Im}\,z}{\partial\,\mathrm{Re}\,\zeta} & \dfrac{\partial\,\mathrm{Im}\,z}{\partial\,\mathrm{Im}\,\zeta} \end{vmatrix}.$$

Our final goal consists of an explicit construction of the function $J(\zeta)$.

Comments.

(A) The Jacobian $J(\zeta)$ has singularities ($J(\zeta) = \infty$) at all branching points—vertices A_ζ, B_ζ, A'_ζ, C_ζ and their images.

(B) One can easily show that all the images of the zero-angled rectangle $A_\zeta B_\zeta A'_\zeta C_\zeta$ in the complex plane ζ have the same area. Namely, the area of the elementary cell (the rectangle $A_z B_z A'_z C_z$) tessellating the plane z is

$$S = \int_{A_z B_z A'_z C_z} \mathrm{d}z\, \mathrm{d}z^*.$$

Performing the transform $z(\zeta)$ one can rewrite S as follows:

$$S = \int_{A_\zeta B_\zeta A'_\zeta C_\zeta} J(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}\zeta^*.$$

Taking into account the correspondence of the boundaries under conformal transforms one arrives at the conclusion that all zero-angled rectangles have the same area S in ζ.

2.2. *Construction of conformal maps*

To find the conformal transform of the square $A_z B_z A'_z C_z$ in the z plane to the zero-angled square $A_\zeta B_\zeta A'_\zeta C_\zeta$ in the ζ plane, consider the following sequence of auxiliary conformal maps (shown in figure 3):

$$z \overset{z=z(w)}{\longrightarrow} w \overset{w=w(\chi)}{\longrightarrow} \chi \overset{\chi=\chi(\zeta)}{\longrightarrow} \zeta.$$

The explicit construction of the functions $z(w)$, $w(\chi)$ and $\chi(\zeta)$ is described below.

(1) The function $z = z(w)$ conformally maps the interior of the triangle $A_z B_z C_z$ with angles $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ in the plane z onto the upper half-plane w (see figure 3(*a*)). The conformal map $z = z(w)$ is performed via Christoffel–Schwartz integral [8]

$$z(w) = \int_0^w \frac{\mathrm{d}\tilde{w}}{\tilde{w}^{1/2}(1-\tilde{w})^{3/4}} \tag{1}$$

with the following correspondence of branching points (see figure 3(*a*)):

$$A_z(z=0) \to A_w(w=0)$$
$$B_z(z=1) \to B_w(w=1)$$
$$C_z(z=i) \to C_w(w=\infty).$$

(2) The function $w = w(\chi)$ conformally maps the upper half-plane Im $w > 0$ to the interior of a circular triangle $A_\chi B_\chi C_\chi$ with angles $(0, 0, 0)$ lying in the upper half-plane Im $\chi > 0$, as is shown in figure 3(*b*). The standard theory of automorphic functions answers the question of explicitly constructing the map $w = w(\chi)$ (see, for example, [9, 10]).
Consider a function $u(w)$ satisfying the hypergeometric equation with three branching points at $w = \{0, 1, \infty\}$:

$$w(w-1)u''(w) + \{(\alpha+\beta+1)w - \gamma\}u'(w) + \alpha\beta\, u(w) = 0 \tag{2}$$

where the coefficients α, β, γ are uniquely defined by the angles of the circular triangle ABC: $\{\mu_1\pi,\ \mu_2\pi,\ \mu_3\pi\}$:

$$\mu_1 = 1-\gamma \qquad \mu_2 = \gamma - \alpha - \beta \qquad \mu_3 = \beta - \alpha.$$

In our case $\mu_1 = \mu_2 = \mu_3 = 0$, hence

$$\alpha = \tfrac{1}{2} \qquad \beta = \tfrac{1}{2} \qquad \gamma = 1.$$

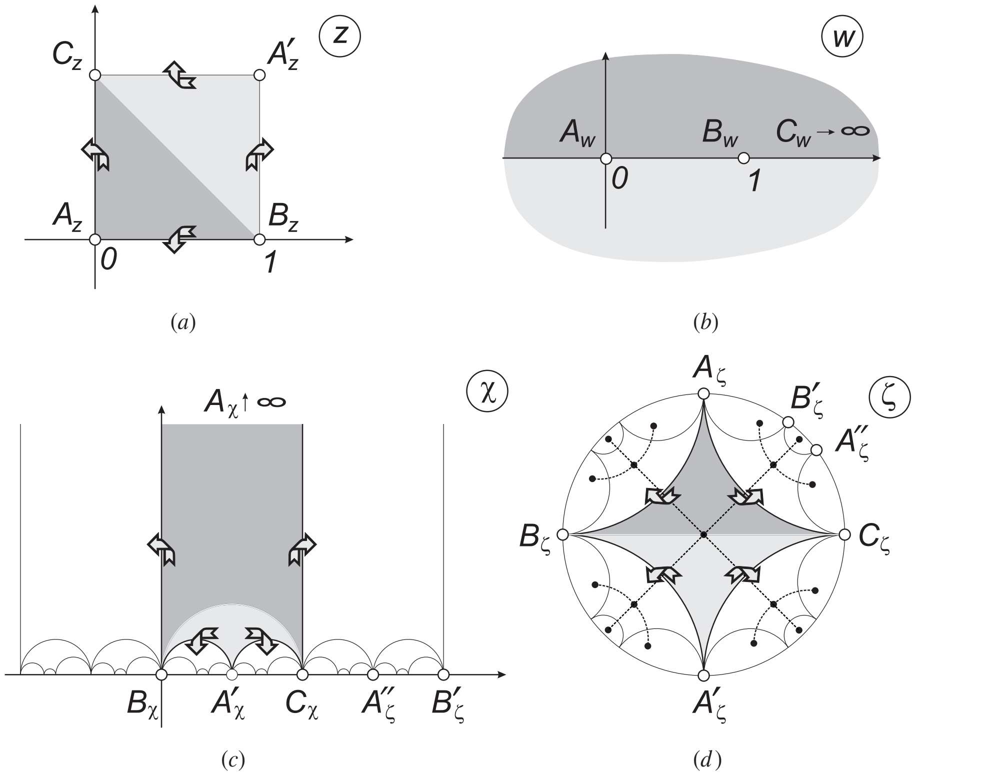


Figure 3. Conformal maps: (a), (b) $z = z(w)$; (b), (c) $w = w(\chi)$; (c), (d) $\chi = \chi(\zeta)$; (d) the Cayley tree isometrically covering the Poincaré disc $|\zeta| < 1$ is shown by a dotted line.

Equation (2) has two linearly independent fundamental solutions $u_1(w)$ and $u_2(w)$ which can be expressed in terms of hypergeometric functions $F(\alpha, \beta, \gamma, z)$. For our particular choice of parameters α, β, γ equation (2) belongs to the so-called degenerate case, where

$$\begin{aligned} u_1(w) &= F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1, z\right) \\ u_2(w) &= \mathrm{i}F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1 - z\right). \end{aligned} \tag{3}$$

Recall that the function $F(\alpha, \beta, \gamma, z)$ admits the following integral representation:

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1}(1 - zt)^{-\alpha}\, \mathrm{d}t.$$

It is known [9–11] that the conformal map $\chi(w)$ of the interior of a circular triangle $A_\chi B_\chi C_\chi$ with angles (0, 0, 0) lying in the upper half-plane Im $\chi > 0$ to the upper half-plane Im $w > 0$ can be written as the quotient of fundamental solutions $u_1(w)$ and $u_2(w)$:

$$\chi(w) = \frac{u_1(w)}{u_2(w)} = -\mathrm{i}\frac{F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right)}. \tag{4}$$

The conformal map of the circular triangle $A_\chi B_\chi C_\chi$ to the upper half-plane Im $w > 0$ is mutually single-valued: thus the inverse function $\chi^{-1}(w)$ solves our problem. Inverting the function $\chi(w)$ (this inverse function satisfies a Schartzian equation, which can be solved in our case, as shown in [11, p 23]), we get

$$w(\chi) = \frac{\vartheta_2^4(0, \mathrm{e}^{\mathrm{i}\pi\chi})}{\vartheta_3^4(0, \mathrm{e}^{\mathrm{i}\pi\chi})}. \tag{5}$$

The conformal transform (5) establishes the following correspondence of branching points:

$$\begin{aligned} A_w(w=0) &\to A_\chi(\chi=\infty) \\ B_w(w=1) &\to B_\chi(\chi=0) \\ C_w(w=\infty) &\to C_\chi(\chi=1). \end{aligned}$$

(3) The function $\chi = \chi(\zeta)$ conformally maps the interior of the circular triangle $A_\chi B_\chi C_\chi$ in the upper half-plane Im $\chi > 0$ to the interior of a circular triangle A_ζ, B_ζ, C_ζ in the open unit disc $|\zeta| < 1$ (see figure 3(*c*)). It is realized via the fractional transform that

$$\chi(\zeta) = \frac{1-\mathrm{i}}{2}\,\frac{\zeta+1}{\zeta-\mathrm{i}} \tag{6}$$

with the following correspondence of branching points:

$$\begin{aligned} A_\chi(\chi=\infty) &\to A_\zeta(\zeta=\mathrm{i}) \\ B_\chi(\chi=0) &\to B_\zeta(\zeta=-1) \\ C_\chi(\chi=1) &\to C_\zeta(\zeta=1). \end{aligned}$$

Collecting equations (1), (5) and (6) we arrive at a composite conformal map:

$$z(\zeta) = z\{w[\chi(\zeta)]\}.$$

The Jacobian $J(\zeta)$ of the map $z(\zeta)$ is

$$\begin{aligned} J(\zeta) &\equiv \left|\frac{\mathrm{d}z(\zeta)}{\mathrm{d}\zeta}\right|^2 = \left|\frac{\mathrm{d}z(w)}{\mathrm{d}w}\right|^2 \left|\frac{\mathrm{d}w(\chi)}{\mathrm{d}\chi}\right|^2 \left|\frac{\chi(\zeta)}{\mathrm{d}\zeta}\right|^2 \\ &= \frac{4}{\pi^2\left|\zeta-\mathrm{i}\right|^4}\left|\vartheta_1'\left(0, \mathrm{e}^{\mathrm{i}\pi\frac{1-\mathrm{i}}{2}\frac{\zeta+1}{\zeta-\mathrm{i}}}\right)\right|^2 \left|\vartheta_2\left(0, \mathrm{e}^{\mathrm{i}\pi\frac{1-\mathrm{i}}{2}\frac{\zeta+1}{\zeta-\mathrm{i}}}\right)\right|^2 \end{aligned} \tag{7}$$

where (see [12])

$$\begin{aligned} \vartheta_1(\tau, \mathrm{e}^{\mathrm{i}\pi\chi}) &= 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}\chi}\sum_{n=0}^{\infty}(-1)^n \mathrm{e}^{\mathrm{i}\pi n(n+1)\chi}\sin(2n+1)\tau \\ \vartheta_1'(0, \mathrm{e}^{\mathrm{i}\pi\chi}) &\equiv \left.\frac{\vartheta_1(\tau, \mathrm{e}^{\mathrm{i}\pi\chi})}{\mathrm{d}\tau}\right|_{\tau=0} = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}\chi}\sum_{n=0}^{\infty}(-1)^n(2n+1)\mathrm{e}^{\mathrm{i}\pi n(n+1)\chi} \\ \vartheta_2(0, \mathrm{e}^{\mathrm{i}\pi\chi}) &= 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}\chi}\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}\pi n(n+1)\chi}. \end{aligned}$$

2.3. *Geometric structure of the leaf's boundary*

Let us parametrize the 'coefficient of deformation' $J(\eta) \equiv J(\eta, \varphi)$ by the variables (η, φ), where η and φ are correspondingly the hyperbolic distance and the polar angle in the unit disc:

$$\eta = \ln\frac{1+|\zeta|}{1-|\zeta|} \qquad \varphi = \arctan\frac{\mathrm{Im}\ \zeta}{\mathrm{Re}\ \zeta}.$$

A few sample plots of the function $J(\eta, \varphi)$ for $0 \leqslant \eta \leqslant \eta_{\max}$ and $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ for $\eta_{\max} = 1.1;\ 1.4;\ 1.7$ are shown in figures 4(*a*)–(*c*).

Equation (7) solves the problem of embedding a 'jupe à godets' isometrically covered by a four-branching Cayley tree into the three-dimensional Euclidean space. The corresponding boundary profile is shown in figure 4.

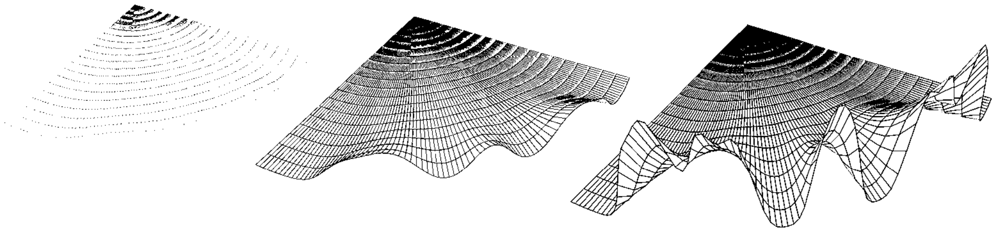


Figure 4. Sample plots of $J(\eta, \varphi)$ for $0 \leqslant \eta \leqslant \eta_{\max}$ and $0 \leqslant \varphi \leqslant \frac{\pi}{2}$: (*a*) $\eta_{\max} = 1.1$; (*b*) $\eta_{\max} = 1.4$; (*c*) $\eta_{\max} = 1.7$.

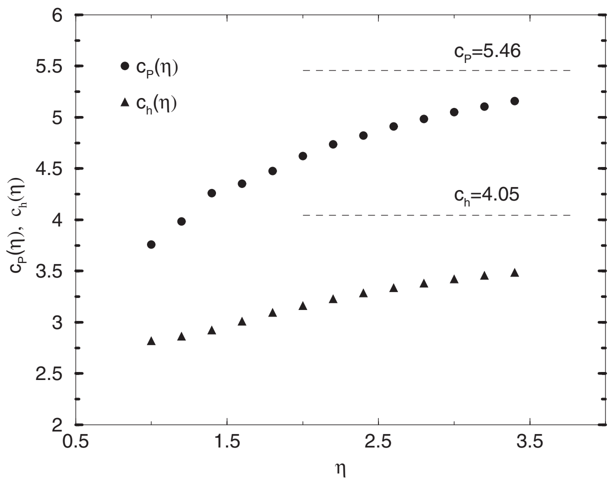


Figure 5. Growth exponents $c_P(\eta)$ and $c_h(\eta)$.

The growth of the perimeter $P(\eta)$ of the circular 'SG' of hyperbolic radius η, as well as the span $h(\eta)$ of transversal fluctuations, can be characterized by the coefficients of growth ('Lyapunov exponents') c_P and c_h, defined as follows:

$$c_P \equiv \lim_{\eta\to\infty} c_P(\eta) = \lim_{\eta\to\infty} \frac{\ln P(\eta)}{\eta} \qquad c_h \equiv \lim_{\eta\to\infty} c_h(\eta) = \lim_{\eta\to\infty} \frac{\ln h(\eta)}{\eta}$$

where

$$P(\eta) = \int_0^{2\pi} \sqrt{1 + \left[\frac{\mathrm{d}J(\eta,\varphi)}{\mathrm{d}\varphi}\right]^2}\, \mathrm{d}\,\varphi$$

and

$$h(\eta) = \int_0^{2\pi} J(\eta,\varphi)\, \mathrm{d}\varphi.$$

The plots of the functions $c_P(\eta)$ and $c_h(\eta)$ are shown in figure 5.

3. Growth of a random 'surface à godets'

The natural generalization of the model of a regular 'SG' defined in section 1 consists in its 'randomization'. Suppose now that the growth phenomenon is random: each extra area is generated only at randomly chosen ('active') sites. It means that, in our model, we insert extra triangles with probability $1-q$ and leave an elementary domain unchanged with probability q. The vertices without an inserted extra triangle we shall call 'defects'. They can be interpreted as defects of curvature, i.e. points where the surface is locally flat. It is easy to understand that, if the probability of defects is $q = 0$, we return to the regular SG with hyperbolic metric, while for $q = 1$ the resulting surface is flat (i.e. it can be embedded in a plane). We are interested in the dependence of the perimeter $P(k|q)$ on the radial distance k for the SG with a concentration of defects q.

Here again the problem becomes much more transparent if formulated for a discrete realization (isometrically embedded graph) of the 'SG'. The issue is to define defects of curvature for graphs. First of all, let us recall the definition of the Cayley graph of a group G.

By definition, the graph of the group G is constructed as follows:

- The vertices of the graph are labelled by group elements. Every element of the group is represented by some irreducible (i.e. written with the minimal number of letters) word (not necessarily unique) built from the letters—generators of the group G. Two words equivalent in the group G correspond to one and the same vertex of the graph.
- Two different vertices corresponding to nonequivalent words w and w' are connected by an edge (i.e. are nearest neighbours) if and only if the word w' can be obtained from the word w by deleting one letter (generator of the group G), or adding one extra letter.

The following construction is of use.

(1) It is known that a four-branching Cayley tree is the graph of the *free* group Γ_2 (see figure 6(*a*)). The group Γ_2 is the infinite group of all possible words constructed from the set of letters $\{g_1, g_2, g_1^{-1}, g_2^{-1}\}$, where there are *no* commutation relations among the letters [6, 13]. The total number of nonequivalent shortest (irreducible) words of length k in the group Γ_2 is equal to the number $P_{\Gamma_2}(k)$ of distinct vertices of the four-branching Cayley tree lying at a distance k from the origin (see figure 6(*a*)):

$$P_{\Gamma_2}(k) = 4 \times 3^{k-1} \qquad (k \geqslant 1). \tag{8}$$

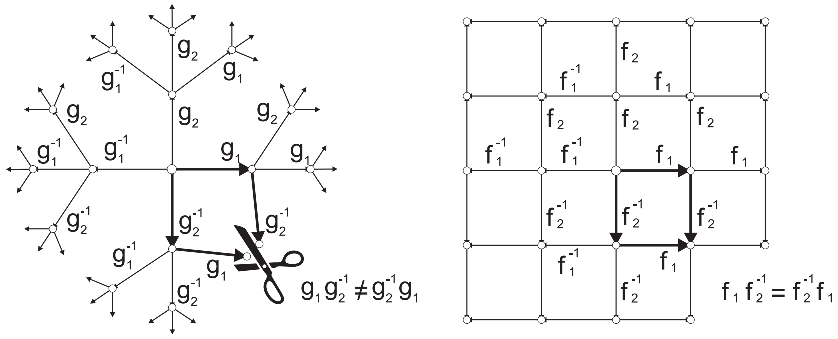
(2) Consider now the opposite case of the *completely commutative group* E_2. This group is generated by $\{f_1, f_2, f_1^{-1}, f_2^{-1}\}$, with the relation $f_1 f_2 = f_2 f_1$. The graph of the group E_2 is a four-vertex lattice isometrically covering the Euclidean plane (see figure 6(*b*)). The problem of comparing two words w_1 and w_2 in the group E_2 written in different ways has a very straightforward solution. Taking into account that all generators of the group E_2 commute, we may write all irreducible words of length k in an *ordered form*:

$$w = g_1^{m_1} g_2^{m_2} \qquad (|m_1| + |m_2| = k)$$

where

$$m_1 = \#[\text{number of generators } f_1] - \#[\text{number of generators } f_1^{-1}]$$
$$m_2 = \#[\text{number of generators } f_2] - \#[\text{number of generators } f_2^{-1}].$$

It is convenient to encode the growth of the number of irreducible words with their length by an 'incident matrix' $\hat{T}$. This construction relies on the automatic structure (see [14]) of the groups under consideration. All words of length $k+1$ are obtained by right hand multiplication of words of length k by admissible generators. The admissible generators for a word w depend *only* on the last letter of w. In our case the entry (i, j) of the matrix $\hat{T}$ is 1 if f_j is admissible after f_i and 0 otherwise. Let us note that the corresponding construction has been used recently



(*a*) (*b*)

Figure 6. (*a*) The four-branching Cayley graph; (*b*) the graph (lattice) covering a surface with a Euclidean (flat) metric.

in [15] in the case of 'locally free groups'. For the commutative group E_2 the incident matrix is

$\hat{T} =$

	f_1	f_2	f_1^{-1}	f_2^{-1}
f_1	1	1	0	1
f_2	0	1	0	0
f_1^{-1}	0	1	1	1
f_2^{-1}	0	0	0	1

. (9)

The total number of shortest words of length k is

$$P(k) = \boldsymbol{v}[\hat{T}]^{k-1}\boldsymbol{v}^{\top} = 4k \qquad (k \geqslant 1) \tag{10}$$

where $\boldsymbol{v} = (1, 1, 1, 1)$ and $\boldsymbol{v}^{\top}$ is the transposed vector.

Our idea to mimic the 'SG' with defects of curvature consists in the following. Let us insert 'defects' in the commutation relations of the free group Γ_2. It means that passing 'along a word' each time one meets a pair of consecutive generators with different subscripts (2 and 1), one either commutes them with probability q, or leaves the sequence without changes with probability $1 - q$. A similar idea has been developed in [16] to approximate braid groups. More explicitly, one makes the following substitutions:

$$\begin{aligned}
g_2\, g_1 &\to \begin{cases} g_1\, g_2 & \text{with probability} \quad q \\ g_2\, g_1 & \text{with probability} \quad 1-q \end{cases} \\
g_2^{-1}\, g_1 &\to \begin{cases} g_1\, g_2^{-1} & \text{with probability} \quad q \\ g_2^{-1}\, g_1 & \text{with probability} \quad 1-q \end{cases} \\
g_2\, g_1^{-1} &\to \begin{cases} g_1^{-1}\, g_2 & \text{with probability} \quad q \\ g_2\, g_1^{-1} & \text{with probability} \quad 1-q \end{cases} \\
g_2^{-1}\, g_1^{-1} &\to \begin{cases} g_1^{-1}\, g_2^{-1} & \text{with probability} \quad q \\ g_2^{-1}\, g_1^{-1} & \text{with probability} \quad 1-q. \end{cases}
\end{aligned} \tag{11}$$

For $q = 0$ we recover the commutation relations of the free group Γ_2 with exponentially growing 'volume' (the number of nonequivalent words) $P_{\Gamma_2}(k|q = 1)$, while for $q = 1$ we arrive at the group E_2 whose 'volume' $P_{E_2}(k|q = 1)$ displays a polynomial growth in k. Thus

$$v_{\Gamma_2} \equiv v(q=0) = \lim_{k\to\infty} \frac{\ln P_{\Gamma_2}(k|q=0)}{k} = \ln 3 > 0$$
$$v_{E_2} \equiv v(q=1) = \lim_{k\to\infty} \frac{\ln P_{E_2}(k|q=1)}{k} = 0.$$

Let us insert 'defects' in the commutation relations of the incident matrix $\hat{T}(x_1, \ldots, x_4)$:

$$\hat{T}(x_1, \ldots, x_4) = \begin{array}{|c|c|c|c|c|} \hline & g_1 & g_2 & g_1^{-1} & g_2^{-1} \\ \hline g_1 & 1 & 1 & 0 & 1 \\ \hline g_2 & x_1 & 1 & x_2 & 0 \\ \hline g_1^{-1} & 0 & 1 & 1 & 1 \\ \hline g_2^{-1} & x_3 & 0 & x_4 & 1 \\ \hline \end{array} \tag{12}$$

where

$$x_m = \begin{cases} 0 & \text{with probability} \quad q \\ 1 & \text{with probability} \quad 1-q \end{cases} \tag{13}$$

and x_m are independent for all $1 \leqslant m \leqslant 4$.

The function $P(k|x_1^{(1)}, \ldots, x_4^{(1)}, \ldots, x_1^{(k)}, \ldots, x_4^{(k)})$, describing the volume growth in an ensemble of words with random commutation relations, is now (compare to (10))

$$P(k|x_1^{(1)}, \ldots, x_4^{(1)}, \ldots, x_1^{(k)}, \ldots, x_4^{(k)}) = \boldsymbol{v}\left[\prod_{i=1}^{k-1} \hat{T}(x_1^{(i)}, \ldots, x_4^{(i)})\right]\boldsymbol{v}^{\top}. \tag{14}$$

We shall distinguish between two distribution of defects: (a) annealed and (b) quenched. In case (a) the partition function $P(k|x_1^{(1)}, \ldots, x_4^{(1)}, \ldots, x_1^{(k)}, \ldots, x_4^{(k)})$ is averaged with the measure (13), while in case (b) the 'free energy', i.e. the logarithmic volume $v \sim \ln P$, is averaged with the same measure.

In the rest of this paper we will pay attention to case (a) only. Averaging the function $P(k|x_1^{(1)}, \ldots, x_4^{(1)}, \ldots, x_1^{(k)}, \ldots, x_4^{(k)})$ over the disorder we get

$$P(k|q) \equiv \langle P(k|x_1^{(1)}, \ldots, x_4^{(1)}, \ldots, x_1^{(k)}, \ldots, x_4^{(k)})\rangle = \boldsymbol{v}\langle \hat{T}(x_1, \ldots, x_4)\rangle^{k-1}\boldsymbol{v}^{\top} \tag{15}$$

where

$$\langle \hat{T}(x_1, \ldots, x_4)\rangle = \sum_{x_1=0}^{1} \cdots \sum_{x_4=0}^{1} q^{4-(x_1+x_2+x_3+x_4)}(1-q)^{x_1+x_2+x_3+x_4}\hat{T}(x_1, x_2, x_3, x_4)$$
$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1-q & 1 & 1-q & 0 \\ 0 & 1 & 1 & 1 \\ 1-q & 0 & 1-q & 1 \end{pmatrix}. \tag{16}$$

The highest eigenvalue $\lambda_{\max}$ of the averaged matrix $\langle \hat{T}(x_1, \ldots, x_4)\rangle$ defines the logarithmic volume $v^a(q) = \lim_{k\to\infty} \frac{\ln P(k|q)}{k}$ of the system with 'annealed' defects in the commutation relations:

$$v^a(q) = \ln\left(1 + 2\sqrt{1-q}\right). \tag{17}$$

One can check that, for any $0 \leqslant q < 1$, the system is characterized by an exponential growth, i.e. $v^a(q) > 0$. However, at $q = q_{\text{cr}} = 1$ one has a transition to a flat metric with $v^a(q=1) = 0$. Expanding $v^a(q)$ in the vicinity of the transition point $q_{\text{cr}} = 1$, one gets

$$v^a(q)\big|_{q\to 1^-} = 2\,(1-q)^{1/2} + \mathrm{O}[(1-q)^{3/2}]. \tag{18}$$

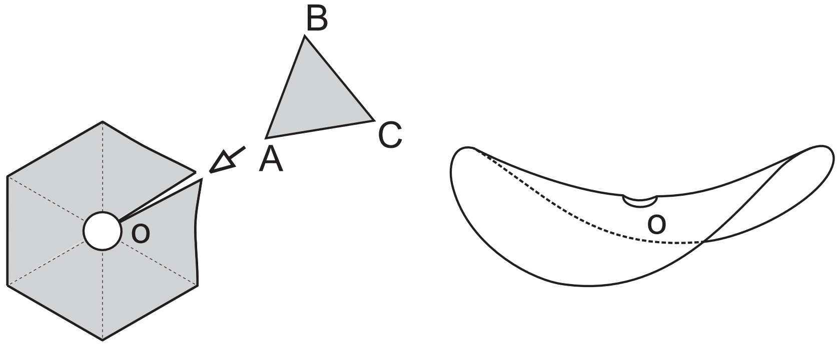


Figure 7. A hexagon with a radial cut and inserted extra triangle and a saddle-like structure near the corner point.

In the framework of the standard classification scheme, the phase transitions are distinguished by the scaling exponent α in the behaviour of the free energy $F(\tau)$ near the critical point τ_{cr} [17]:

$$F(\tau)\big|_{\tau\to\tau_{\rm cr}} \propto |\tau_{\rm cr} - \tau|^{\alpha}. \tag{19}$$

Comparing (19) and (18) one can attribute (18) to a phase transition with the critical exponent $\alpha = \frac{1}{2}$.

4. Discussion

In sections 1 and 2 we have developed a method based on conformal transforms, allowing us to describe the growth of a regular 'jupe à godets'. Let us stress that our derivation is based on the assumption that the 'jupe à godets' is isometrically covered by a four-branching Cayley tree. We do not pretend to cover all possible SG with this construction (let us recall that an exponentially growing structure like a 'jupe à godets' has infinitely many discrete lattices of isometries), but to provide an analytical tool for a quantitative description of the main features of such surfaces. It would be, in a sense, more natural to suggest a construction corresponding to another hyperlattice (a so-called {3, 7} lattice [18]). Take the right hexagon consisting of six equal-sided triangles and make a radial cut along the side of one triangle, as is shown in figure 7. In a given cut, insert an extra triangle such that the central point is surrounded now by seven triangles; the resulting structure can be tentatively denoted as the 'right 7-gon'. It is clear that such a structure cannot be embedded in a plane any longer and locally (near the centre of a 7-gon) has a saddle-like structure—see figure 7.

Let us continue the surface construction and glue new equal-sided triangles to all perimeter bonds of a given 7-gon in such a way that each corner of any triangle in the surface is surrounded by exactly 7 triangles. Hence, near each corner point the surface has a saddle-like structure. In what follows we shall call this structure the '7-gon SG'.

Consider now the unit disc endowed with the hyperbolic Poincaré metric and take the equal-sided circular (i.e. bounded by geodesics) triangle ABC with angles $\left\{\frac{2\pi}{7}, \frac{2\pi}{7}, \frac{2\pi}{7}\right\}$ as shown in figure 8.

Make reflections of the interior of the triangle ABC with respect to its sides and get the images of the initial triangle, then make reflections of the images with respect to their own sides and so on. In this way one tessellates the whole disc with the images of the initial triangle ABC. It is easy to establish a topological bijection between the model of glued 7-gons and the

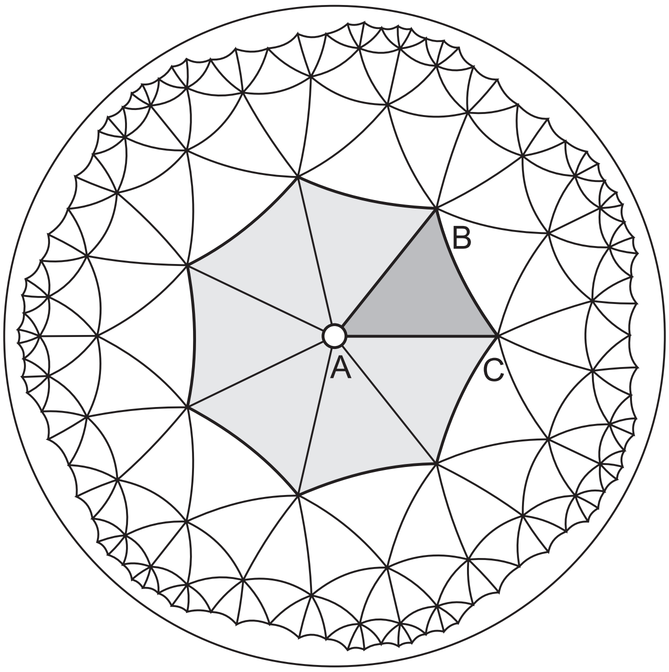


Figure 8. A tessellation of a Poincaré disc by triangles with angles $\left\{\frac{2\pi}{7},\ \frac{2\pi}{7},\ \frac{2\pi}{7}\right\}$.

system of hyperbolic triangles with angles $\left\{\frac{2\pi}{7},\ \frac{2\pi}{7},\ \frac{2\pi}{7}\right\}$, because all vertices are surrounded by exactly 7 curvilinear hyperbolic triangles—see figure 8.

The structure shown in figure 8 is invariant under conformal transforms of the Poincaré disc onto itself :

$$\zeta \to \frac{\zeta - \zeta_0}{\zeta \zeta_0^* - 1}$$

where ζ and ζ^* are complex-conjugated and ζ_0^* is the coordinate of any image of the point 0 (the vertex A of triangle ABC). Such a conformal transform allows us to shift any image of a vertex of any triangle to the centre 0 of the disc (hence, such a conformal transform replaces a translation in Euclidean geometry).

It would be desirable to develop the symbolic language allowing us to investigate the statistical properties of a 'SG' with and without random 'defects of curvature' for the structure shown in figure 8. We expect that a symbolic language similar to the one developed in section 3 applied to the group of reflections of triangles with angles $\left\{\frac{2\pi}{7},\ \frac{2\pi}{7},\ \frac{2\pi}{7}\right\}$ would allow such a generalization.

Let us conclude with the following remark. The conformal embedding of a '7-gon SG' shown in figure 8 into a 3D Euclidean space can be algorithmically solved in the same way as has been done for the four-branching Cayley tree. However, for the '7-gon surface' we cannot derive an explicit expression for the coefficient of deformation $J(\zeta)$ in a simple form (as in equation (7)). We outline this construction following the results of section 2. Let $\tilde{z}(\tilde{\zeta})$ be the conformal transform of the triangle $\tilde{A}_z\tilde{B}_z\tilde{C}_z$ with angles $\left\{\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3}\right\}$ in the complex plane z to the triangle $\tilde{A}_\zeta\tilde{B}_\zeta\tilde{C}_\zeta$ with angles $\left\{\frac{2\pi}{7},\ \frac{2\pi}{7},\ \frac{2\pi}{7}\right\}$ in the Poincaré disc $|\zeta| < 1$. The Jacobian $J(\tilde{\zeta})$ solves the problem of isometric embedding of a '7-gon SG' into the 3D Euclidean space. For the proper choice of parameters α, β, γ in (2) we can derive the fundamental solutions

$u_1(w)$ and $u_2(w)$ (and, hence, $\chi(w)$). However, unfortunately we are unable to find the inverse function $w(\chi)$. An explicit expression for $J(\tilde{\zeta})$ is therefore out of reach so far.

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[6] SN would like to thank Olga Yartseva for help in finding the name for the surface under consideration.